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Continuous location problems and Big Triangle Small Triangle: constructing better bounds

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Abstract The Big Triangle Small Triangle method has shown to be a powerful global optimization procedure to address continuous location problems. In the paper published in *J. Global Optim.* (37:305–319, 2007), Drezner proposes a rather general and effective approach for constructing the bounds needed. Such bounds are obtained by using the fact that the objective functions in continuous location models can usually be expressed as a difference of convex functions. In this note we show that, exploiting further the rich structure of such objective functions, alternative bounds can be derived, yielding a significant improvement in computing times, as reported in our numerical experience.

Keywords Continuous location · Big Triangle Small Triangle · Dc monotonic functions

1 Introduction

In continuous location problems, the location for one or several facilities within a subset S of the n-dimensional space $\mathbb{R}^n$ is sought so that a given function of the distances from the facilities to a set A of users is optimized. The reader is referred to Plastria [9] for an introduction to continuous location.

Many instances in single-facility continuous location can be expressed as optimization problems in the form

$$\min_{x \in S} F(x) := \sum_{a \in A} \varphi_a(\|x - a\|_a) \tag{1.1}$$

where S is a finite union of polytopes in $\mathbb{R}^n$ representing the set of possible locations for the facility, A is a finite subset of $\mathbb{R}^n$ with the coordinates of the users, $\|\cdot\|_a$ is a norm in $\mathbb{R}^n$ for

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each $a \in A$ which models travel distances from user a, and φ_a is a function, $\varphi_a : \mathbb{R}_+ \longrightarrow \mathbb{R}$ so that $\varphi_a(d)$ gives the cost associated with the interaction between user a and the facility located at distance d.

In general F is not convex, and global optimization procedures are needed to solve (1.1). The first solution method proposed in the literature was a branch-and-bound algorithm called Big Square Small Square, BSSS [7], later generalized by Plastria [8]. Recently, Drezner and Suzuki have introduced in [5] a variant, called the Big Triangle Small Triangle, BTST. BTST differs from its ancestor BSSS in the subdivision elements used: whereas BSSS uses hyper-rectangles, BTST uses simplices. Both may share the bounding strategies, but in the literature one finds that bounds in BSSS are mostly constructed exploiting the (piecewise) monotonicity of the functions φ_a, whereas in BTST functions φ_a are assumed to be dc, i.e., they can be decomposed as a difference of convex functions, and then standard bounding procedures for dc functions are then used. See [3,4,6] for some examples.

In most applications, such as all those mentioned in Drezner [2], functions φ_a are dc, and a dc decomposition of φ_a as $\varphi_a = \varphi_a^1 - \varphi_a^2$ is available. This immediately yields lower and upper bounds for φ_a. Indeed, an upper bound of a univariate finite convex function on a segment is given by the chord interpolating at the endpoints; a lower bound is obtained by taking the supporting line at an arbitrary interior point. Using this strategy, given an interval $[d_a^{\min}, d_a^{\max}] \subset \mathbb{R}_+$, one constructs coefficients $K_a^1, L_a^1, K_a^2, L_a^2$, such that, for any $d \in [d_a^{\min}, d_a^{\max}]$,

$$\begin{aligned} \varphi_a^1(d) &\geq K_a^1 d + L_a^1 \\ \varphi_a^2(d) &\leq K_a^2 d + L_a^2, \end{aligned} \tag{1.2}$$

implying

$$\varphi_a(d) \geq (K_a^1 - K_a^2)d + (L_a^1 - L_a^2). \tag{1.3}$$

Hence, for any $x \in \mathbb{R}^n$ such that $d_a^{\min} \leq \|x - a\|_a \leq d_a^{\max}$, we have by (1.3) that

$$\varphi_a(\|x - a\|_a) \geq (K_a^1 - K_a^2)\|x - a\|_a + (L_a^1 - L_a^2). \tag{1.4}$$

A concave minorant $l_a(\cdot)$ of $\varphi_a(\| \cdot - a\|_a)$, i.e., a concave function with $l_a(x) \leq \varphi_a(\|x - a\|_a)\, \forall x$, is obtained as follows. If $K_a^1 - K_a^2 \geq 0$, then the function in the right term of (1.4) is convex, a concave minorant of which is obtained by linearizing below the convex function $(K_a^1 - K_a^2)\|x - a\|_a + (L_a^1 - L_a^2)$. On the other hand, if $K_a^1 - K_a^2 < 0$, then the function in the right term of (1.4) is concave, which is obviously a concave minorant of itself.

With this, we can construct a concave minorant $m_a(x)$ on a simplex (triangle if $n = 2$) of $\varphi_a(\|x - a\|_a)$, and, by summing such minorants, one obtains a concave minorant of F on the simplex. A lower bound of F is thus obtained by inspecting at the extreme points of the simplex such concave minorant.

The aim of this paper is to show that, in many cases, it is possible to obtain a dc decomposition of the functions φ_a with additional properties, namely, that the corresponding φ_a^1, φ_a^2 are not only convex, but also monotonic. Moreover, this decomposition leads to bounding procedures which may be more successful in terms of running times than the standard bounding procedures in the same branch and bound scheme.

The remainder of the paper is structured as follows. In Sect. 2 we introduce a subclass of dc functions. Some general properties are studied, and different examples (in the context of continuous location problems) are given. In Sect. 3 a bounding strategy for dcm functions is given, which is tested in a set of numerical examples in Sect. 4, showing that this new strategy is very competitive in running times.

2 Dcm functions

2.1 General properties

Definition 1 Given a nondegenerate interval $K \subset \mathbb{R}$, a function $\varphi : K \longrightarrow \mathbb{R}$ is said to be difference of convex monotonic (dcm) in K if there exist $\varphi^1, \varphi^2 : K \longrightarrow \mathbb{R}$, convex and monotonic in K such that $\varphi = \varphi^1 - \varphi^2$.

Smooth functions are dcm, as shown in the following results.

Proposition 2 *Let φ be C^2 in a nondegenerate interval $K \subset \mathbb{R}$. Then φ is dcm in K.*

Proof Let t_0 be interior to K. Define

$$\alpha(t) = \int_{t_0}^{t} \left[\varphi''(s)\right]^{+} \mathrm{d}s$$
$$\beta(t) = -\int_{t_0}^{t} \left[\varphi''(s)\right]^{-} \mathrm{d}s,$$

where $[z]^+$ and $[z]^-$ denote, respectively, the positive and negative part of z, that is, $[z]^+ = \max\{z, 0\}$ and $[z]^- = \min\{z, 0\}$. We have the following decomposition for φ :

$$\varphi(t) = \int_{t_0}^{t} \alpha(s)\,\mathrm{d}s + (\varphi(t_0) + \varphi'(t_0)(t - t_0)) - \int_{t_0}^{t} \beta(s)\,\mathrm{d}s.$$

If $\varphi'(t_0) \geq 0$, then both $\int_{t_0}^{t} \alpha(s)\,\mathrm{d}s + \varphi(t_0) + \varphi'(t_0)(t - t_0)$ and $\int_{t_0}^{t} \beta(s)\,\mathrm{d}s$ are convex and non-decreasing, whereas if $\varphi'(t_0) < 0$, then $\int_{t_0}^{t} \alpha(s)\,\mathrm{d}s$ and $-\varphi(t_0) - \varphi'(t_0)(t - t_0) + \int_{t_0}^{t} \beta(s)\,\mathrm{d}s$ are convex and non-decreasing, giving a dcm decomposition of φ. □

Proposition 3 *Let K be a compact interval. Assume $\varphi = \varphi^1 - \varphi^2$, with both $\varphi^1, \varphi^2 \in C^1(K)$ and convex in K. Then, φ is dcm in K.*

Proof Let $M \geq \max\{\max_{t \in K} -(\varphi^1(t))', \max_{t \in K} -(\varphi^2(t))'\}$. Then

$$\begin{aligned} \varphi(t) &= \varphi^1(t) - \varphi^2(t) \\ &= (\varphi^1(t) + tM) - (\varphi^2(t) + tM), \end{aligned}$$

giving a dcm decomposition for φ. □

Remark 4 Although the class of dcm functions is rather broad, it is a proper subset of the class of dc functions. Indeed, let $K = [0, 1]$, and consider the function $\varphi(t) = \sqrt{t(1-t)}$, which is concave in K, and thus dc in K. Let us show that φ is not dcm. By contradiction, suppose φ is dcm in K, and let $\varphi = \varphi^1 - \varphi^2$ be a dcm decomposition in K. Since the side derivative of φ at $t = 0$ is $+\infty$, the right derivative of φ^1 at $t = 0$ is also $+\infty$, or the right derivative of φ^2 at $t = 0$ is $-\infty$. In the former case, the convexity of φ^1 would imply that its right derivative would be non-decreasing, and thus equally constant to $+\infty$ in K, which is a contradiction. In the latter case, φ^2 would be non-increasing in K. Since the left derivative of φ at $t = 1$ is $-\infty$, we would need that the left derivative of φ^1 is also $-\infty$ (impossible, by convexity of φ^1), or the left derivative of φ^2 at $t = 1$ would be $+\infty$, which contradicts the fact that φ^2 is non-increasing in K. Hence, no dcm decomposition for φ exists.

Remark 5 Contrary to the case of dc functions, which enjoy a rich algebra (dc functions are closed under usual operations), the class of dcm functions is not closed by sums. This is shown with the following example: take $K = \mathbb{R}_+$, and consider the dcm functions in K

$$\alpha(t) = \sqrt{t} = 0 - \left(-\sqrt{t}\right)$$
$$\beta(t) = -e^t = 0 - e^t.$$

The function $\varphi(t) = \alpha(t) + \beta(t)$ is not dcm in K. Indeed, suppose by contradiction that it is dcm, and a dcm decomposition is given by $\varphi = \varphi^1 - \varphi^2$. As in Remark 4, let us analyze the directional derivatives at the endpoints of K. Since the right derivative $\varphi'_+(0)$ of φ at $t = 0$ is $+\infty$, by convexity of φ^1 and φ^2 we would have that φ^2 should have right derivative at $t = 0$ equal to $-\infty$, which would imply, in particular, that φ^2 would be non-increasing in K. Now, for sufficiently large t, the derivative of φ goes to $-\infty$. Hence, since φ^2 is non-increasing, one would need φ^1 to have derivative going to $-\infty$, which is impossible by convexity of φ^1.

2.2 Examples in continuous location

Although the theory of dcm functions is general, we show that it is directly applicable, among others, to continuous location problems. In what follows we describe some of the models already presented in Drezner [2], which fit within the framework given in this paper.

2.2.1 Obnoxious facility location

$$\min_x \sum_{a\in A} \frac{\omega_a}{\|x-a\|^2} \quad (\omega_a > 0 \quad \forall a)$$

A dcm decomposition in $\mathbb{R}_{++}$ for $\varphi_a(d) = \omega_a/d^2$ is given by

$$\varphi_a^1(d) = \omega_a/d^2 \quad \varphi_a^2(d) = 0 \tag{2.1}$$

2.2.2 Weber problem with some negative weights

$$\min_x \sum_{a\in A} \omega_a \|x-a\| \quad (\omega_a \in \mathbb{R})$$

A dcm decomposition in $\mathbb{R}_+$ for $\varphi_a(d) = \omega_a d$ is given by

$$\varphi_a^1(d) = \max\{\omega_a, 0\}d \quad \varphi_a^2(d) = \max\{-\omega_a, 0\}d \tag{2.2}$$

2.2.3 Huff competitive location

$$\max_x \sum_{a\in A} \frac{b_a}{1 + h_a\|x-a\|^\lambda} \quad (h_a, b_a > 0 \quad \lambda \geq 1)$$

A dcm decomposition in $\mathbb{R}_+$ of $\varphi_a(d) = -\frac{b_a}{1+h_a d^\lambda}$ is given by

$$\begin{aligned} \varphi_a^1(d) &= b_a h_a d^\lambda \\ \varphi_a^2(d) &= b_a h_a d^\lambda - \varphi_a(d) \end{aligned} \tag{2.3}$$

An alternative dcm decomposition for φ_a is given by

$$\varphi_a^1(d) = \begin{cases} -\varphi_a(\bar{d}) - \varphi_a'(\bar{d})(d-\bar{d}) + \varphi_a(d) & \text{if } d < \bar{d} \\ 0 & \text{if } d \geq \bar{d} \end{cases}$$
$$\varphi_a^2(d) = \begin{cases} -\varphi_a(\bar{d}) - \varphi_a'(\bar{d})(d-\bar{d}) & \text{if } d < \bar{d} \\ -\varphi_a(d) & \text{if } d \geq \bar{d} \end{cases} \tag{2.4}$$

In these expressions, $\bar{d}$ is root of equation $\varphi_a''(d) = 0$.

2.2.4 Stochastic weighted minimax

$$\max_x \sum_{a \in A} \ln(\|x-a\|) - \ln(T_a - \alpha_a \|x-a\|)$$

A dcm decomposition in $\mathbb{R}_{++}$ for $\varphi_a(d) = -\ln(d) + \ln(T_a - \alpha_a d)$ is given by

$$\varphi_a^1(d) = -\ln(d), \quad \varphi_a^2(d) = -\ln(T_a - \alpha_a d). \tag{2.5}$$

2.2.5 Unserviced demand (I)

$$\max_x \sum_{a \in A} \exp(-\|x-a\|)$$

As dcm decomposition in $\mathbb{R}_+$ for $\varphi_a(d) = -\exp(-d)$ we have

$$\begin{aligned} \varphi_a^1(d) &= 0 \\ \varphi_a^2(d) &= \exp(-d). \end{aligned} \tag{2.6}$$

2.2.6 Unserviced demand (II)

$$\max_x \sum_{a \in A} \frac{1}{1+\|x-a\|}$$

We have as dcm decomposition in $\mathbb{R}_+$ for $\varphi_a(d) = -1/(1+d)$

$$\begin{aligned} \varphi_a^1(d) &= 0 \\ \varphi_a^2(d) &= 1/(1+d) \end{aligned} \tag{2.7}$$

2.2.7 Inventory-location model

$$\min_x \sum_{a \in A} \alpha_a \|x-a\| + \omega_a \sqrt{R_a \|x-a\|^2 + S_a \|x-a\| + T_a}$$

A dcm decomposition in $\mathbb{R}_+$ for $\varphi_a(d) = \omega_a \sqrt{R_a d^2 + S_a d + T_a}$ is given by

$$\begin{aligned} \varphi_a^1(d) &= 0 \\ \varphi_a^2(d) &= -\alpha_a d - \omega \sqrt{R_a d^2 + S_a d + T_a} \end{aligned} \tag{2.8}$$

2.2.8 Gradual covering

$$\min_x \sum_{a \in A} \phi_a(x),$$

with

$$\phi_a(x) := \begin{cases} 0 & \text{if } \|x-a\| \le l_a \\ \omega_a(\|x-a\| - l_a) & \text{if } l_a < \|x-a\| \le u_a \\ \omega_a(u_a - l_a) & \text{if } \|x-a\| > u_a \end{cases}$$

A dcm decomposition is given by

$$\begin{aligned} \varphi_a^1(d) &= \begin{cases} 0 & \text{if } d < l_a \\ \omega_a(d-l_a) & \text{if } d \ge l_a \end{cases} \\ \varphi_a^2(d) &= \begin{cases} 0 & \text{if } d < u_a \\ \omega_a(d-u_a) & \text{if } d \ge u_a \end{cases} \end{aligned} \tag{2.9}$$

2.2.9 The acceleration–deceleration distance

$$\min_x \sum_{a \in A} \phi_a(x),$$

where

$$\phi_a(x) = \begin{cases} 2\sqrt{\|x-a\| d_{0a}} & \text{if } \|x-a\| < d_{0a} \\ \|x-a\| + d_{0a} & \text{if } \|x-a\| \ge d_{0a} \end{cases}$$

A dcm decomposition is given by

$$\varphi_a^1(d) = 0 \quad \varphi_a^2(d) = -\begin{cases} 2\sqrt{d d_{0a}} & \text{if } d < d_{0a} \\ d + d_{0a} & \text{if } d \ge d_{0a} \end{cases} \tag{2.10}$$

3 Bounds for dcm functions

Consider, for each $a \in A$, a function $\varphi_a = \varphi_a^1 - \varphi_a^2$, dcm in $\mathbb{R}_+$. Let S be a polytope in $\mathbb{R}^n$, expressed as the convex hull of a finite set of points $\{v_i : i \in I\}$. W.l.o.g. we assume that S contains at least a non-degenerate segment. Let us construct a lower bound in S for $F(x) = \sum_{a \in A} \varphi_a(\|x-a\|_a)$ using the monotonicity of the functions φ_a^1, φ_a^2.

For $x \in S$, we express x as $x = \sum_i \lambda_i v_i$, $\lambda_i \ge 0\, \forall i$, $\sum_i \lambda_i = 1$. We first obtain a concave minorant of φ_a^1 as follows:

(1) If φ_a^1 is non-decreasing, then $\varphi_a^1(\|\cdot - a\|_a)$ is the composition of a non-decreasing convex function with a convex function. Hence, it is also convex. Let $x_0 \in S$, $x_0 \ne a$, and let p_a be a subgradient at x_0 of the convex function $\varphi_a^1(\|\cdot - a\|_a)$. By definition of subgradient we have that

$$\varphi_a^1\left(\left\|\sum_i \lambda_i v_i - a\right\|_a\right) \ge \varphi_a^1\left(\|x_0 - a\|_a\right) + p_a^\top\left(\sum_i \lambda_i v_i - x_0\right).$$

Observe that the minorant found is an affine function.

(2) If φ_a^1 is non-increasing, then given $x_0 \in S$, $x_0 \neq a$, for any p_a, subgradient at $d_0 := \|x_0 - a\|_a$ of φ^1, by definition of subgradient, one has

$$\varphi_a^1(d) \geq \varphi_a^1(d_0) + p_a\,(d - d_0)\,,$$

and then,

$$\varphi_a^1\left(\left\|\sum_i \lambda_i v_i - a\right\|_a\right) \geq \varphi_a^1(\|x_0 - a\|_a) + p_a\left(\left\|\sum_i \lambda_i v_i - a\right\|_a - \|x_0 - a\|_a\right).$$

Since φ^1 is assumed to be non-increasing, one has that $p_a \leq 0$, and hence the minorant found is concave.

Now we obtain a convex majorant of $\varphi_a^2(\|\cdot - a\|_a)$, i.e., a convex function u_a with $u_a(x) \geq \varphi_a^2(\|x - a\|_a)\ \forall x$:

(1) If φ_a^2 is non-decreasing, then $\varphi_a^2(\|\cdot - a\|_a)$ is the composition of a non-decreasing convex function with a convex function. Hence, it is also convex, and the very same function $\varphi_a^2(\|\cdot - a\|_a)$ is taken as convex majorant of itself.

(2) If φ_a^2 is non-increasing, then given $x_0 \in S$, $x_0 \neq a$, let p_a be a subgradient of $\|\cdot - a\|_a$ at x_0. Then, by definition of subgradient, one has

$$\left\|\sum_i \lambda_i v_i - a\right\|_a \geq \|x_0 - a\|_a + p_a^\top\left(\sum_i \lambda_i v_i - x_0\right),$$

and, since φ^2 is non-increasing,

$$\varphi_a^2\left(\left\|\sum_i \lambda_i v_i - a\right\|_a\right) \leq \varphi_a^2\left(\|x_0 - a\|_a + p_a^\top\left(\sum_i \lambda_i v_i - x_0\right)\right).$$

We have then found a convex majorant of $\varphi_a^2(\|\cdot - a\|_a)$.

The procedure above yields a concave minorant $l_a(x)$ of $\varphi_a^1(\|x - a\|_a)$ and a convex majorant $u_a(x)$ of $\varphi_a^2(\|x - a\|_a)$. This implies that the function $l_a(x) - u_a(x)$ is a concave minorant of $\varphi_a(\|\cdot - a\|_a) = \varphi_a^1(\|\cdot - a\|_a) - \varphi_a^2(\|\cdot - a\|_a)$, and hence the concave function $\sum_{a\in A}(l_a(x) - u_a(x))$ is a concave minorant of $F(x)$. This implies that

$$\min_{x\in S} F(x) \geq \min_{x\in S}\sum_{a\in A}(l_a(x) - u_a(x)) = \min_{i\in I}\sum_{a\in A}(l_a(v_i) - u_a(v_i)),$$

and this is the bound we propose.

4 Computational experience

In order to show empirically that the bounding strategy described in the paper is competitive compared with the approach suggested by Drezner, we have implemented the branch and bound method BTST using the two bounding procedures and run the algorithm on a set of instances of the 2-dimensional problems described in Sect. 2.2. The algorithm was implemented in a Fortran program compiled by Intel Fortran 10.1, and run on a 2.4 GHz computer under Windows XP. The solutions were found to a relative accuracy of 10^{-10}.

Table 1 Problems, dcm decompositions and bounding strategies considered

Experiment	Problem name	Dcm decompositions	
		JOGO(2007) bounding method	Dcm-based bounding method
A	Obnoxious facility location	2.1	2.1
B	Weber problem with some negative weights	2.2	2.2
C	Huff competitive location	2.3	2.3
D	Huff competitive location	2.3	2.4
E	Huff competitive location	2.4	2.4
F	Stochastic weighted minimax	2.5	2.5
G	Unserviced demand (I)	2.6	2.6
H	Unserviced demand (II)	2.7	2.7
I	Inventory-location model	2.8	2.8
J	Gradual covering	2.9	2.9
K	The accelaration–deceleration distance	2.10	2.10

Two issues must be taken into account, namely, the dcm decomposition and the bounding process. In Table 1 we show the problems that have been considered in the numerical experience, as well as the dcm decompositions used in the two bounding strategies: the bounding procedure described in Drezner [2], summarized in Sect. 1, and the new procedure, detailed in Sect. 3. The numbers in the last two columns of Table 1 are the labels of the corresponding dc decompositions given in Sect. 2.2. Note that in experiment D (Huff competitive location) two different dc decompositions (namely 2.3 and 2.4) have been used, whereas in the remaining experiments the two bounding methods are compared with respect the same decomposition.

Every problem was solved, using the two bounding procedures, for a different numbers of demand points N, ranging from 10 to 10,000, randomly generated in the unit square $[0, 1] \times [0, 1]$.

The computational results obtained for these problems are shown in Tables 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12. Each table shows some statistics (minimum, maximum and average) for three indicators of the algorithm performance: number of iterations, maximum number of triangles in the branch-and-bound list (we remind that in the BTST method simplices, and thus triangles when the dimension $n = 2$, are used as partition elements) and running time. Every problem was run ten times for each value of N in order to obtain the above-mentioned measurements.

The Huff competitive location problem is analyzed in experiments C, D and E. When the decomposition 2.3 is used, Drezner's method outperforms the dcm-based method. However, when one uses the dc decomposition 2.4, which exploits more the structure of the functions φ_a, the gains in time and memory use of the dcm-based method are very important. Moreover, as shown in Table 5, the decomposition 2.4 combined with our dcm-based method, clearly outperforms the decomposition 2.3 combined with Drezner's method.

In the remaining problems, when the same dc decomposition is used, the two bounding methods yield roughly the same number of iterations and memory use (measured as the maximum number of triangles to be inspected), but our dcm-based method tends to run in much less time (experiments F, G, H, I, J and K) or the same time (experiments A and B).

To sum up, it is evident that in most cases the new bounding procedure suggested in this paper reduces considerably the running times for the same dcm decomposition. The choice of the dcm decomposition may have important consequences, as shown in experiments C–E. An adequate choice of the dc decomposition, following [1], deserves further study.

Table 2 Computational results for experiment A in Table 1

N	Iterations			Max. triangles			Time (s)		
	Min.	Max.	Ave.	Min.	Max.	Ave.	Min.	Max.	Ave.
JOGO(2007) bounding method									
10	37	77	65.50	10	34	17.30	0.00	0.00	0.00
20	71	141	88.00	15	40	27.80	0.00	0.00	0.00
50	76	186	105.40	23	108	50.00	0.00	0.01	0.00
100	98	387	170.10	38	184	80.40	0.00	0.03	0.01
200	103	496	202.00	57	211	133.40	0.01	0.06	0.03
500	115	215	164.30	104	199	157.10	0.09	0.10	0.10
1,000	158	542	317.90	196	614	334.80	0.31	0.54	0.40
2,000	207	1,898	567.20	272	781	527.30	1.12	3.07	1.54
5,000	312	1,984	609.10	399	1,768	710.80	6.42	11.31	7.28
10,000	368	1,703	647.90	623	1,511	986.80	24.10	31.93	25.75
Dcm-based bounding method									
10	37	77	65.50	10	34	17.30	0.00	0.00	0.00
20	71	141	88.00	15	40	27.80	0.00	0.01	0.00
50	76	186	105.40	23	108	50.00	0.00	0.01	0.00
100	98	387	170.10	38	184	80.40	0.00	0.03	0.01
200	103	496	202.00	57	211	133.40	0.01	0.06	0.03
500	115	215	164.30	104	199	157.10	0.09	0.12	0.10
1,000	158	542	317.90	196	614	334.80	0.31	0.54	0.41
2,000	207	1,898	567.20	272	781	527.30	1.15	3.09	1.56
5,000	312	1,984	609.10	399	1,768	710.80	6.53	11.43	7.41
10,000	368	1,703	647.90	623	1,511	986.80	24.68	32.43	26.30

Table 3 Computational results for experiment B in Table 1

N	Iterations			Max. triangles			Time (s)		
	Min.	Max.	Ave.	Min.	Max.	Ave.	Min.	Max.	Ave.
JOGO(2007) bounding method									
10	115	1,712	393.10	12	169	39.90	0.00	0.01	0.00
20	120	825	292.60	12	127	44.20	0.00	0.01	0.00
50	132	5,689	769.00	16	479	88.30	0.00	0.10	0.01
100	124	481	259.00	19	119	41.20	0.00	0.01	0.01
200	181	633	327.50	25	99	49.30	0.01	0.06	0.03
500	269	1,762	598.30	32	775	174.10	0.09	0.40	0.16
1,000	248	760	406.20	27	221	79.80	0.26	0.46	0.32
2,000	371	4,233	888.30	62	1,428	237.80	0.93	4.10	1.36
5,000	389	19,638	4,040.70	74	7,115	1,335.60	4.75	44.25	12.22
10,000	362	13,761	4,218.40	90	4,308	1,382.60	17.28	72.14	33.06
Dcm-based bounding method									
10	115	1,712	393.10	12	169	39.90	0.00	0.01	0.00
20	120	825	292.60	12	127	44.20	0.00	0.01	0.00
50	132	5,689	769.00	16	479	88.30	0.00	0.10	0.01
100	124	481	259.00	19	119	41.20	0.00	0.03	0.01
200	181	633	327.50	25	99	49.30	0.01	0.06	0.03
500	269	1,762	598.30	32	775	174.10	0.09	0.39	0.16
1,000	248	760	406.20	27	221	79.80	0.25	0.45	0.32
2,000	371	4,233	888.30	62	1,428	237.80	0.92	4.06	1.34
5,000	389	19,638	4,040.70	74	7,115	1,335.60	4.68	43.64	12.07
10,000	362	13,761	4,218.40	90	4,308	1,382.60	17.01	71.32	32.63

Table 4 Computational results for experiment C in Table 1

N	Iterations			Max. triangles			Time (s)		
	Min.	Max.	Ave.	Min.	Max.	Ave.	Min.	Max.	Ave.
JOGO(2007) bounding method									
10	1,546	10,272	3,455.40	353	4,384	1,210.60	0.04	0.25	0.08
20	3,261	23,784	10,546.30	942	10,401	4,202.50	0.15	1.17	0.51
50	3,424	26,321	9,037.90	902	3,033	1,787.10	0.42	3.21	1.10
100	8,631	38,740	20,839.70	1,440	15,614	5,968.30	2.10	9.40	5.06
200	11,374	31,149	18,604.70	2,561	7,331	4,418.20	5.56	15.15	9.05
500	25,686	225,775	60,419.40	2,974	56,600	12,151.50	31.34	272.73	73.28
1,000	16,119	121,410	49,726.00	4,060	36,009	9,533.30	40.03	294.57	121.40
2,000	25,003	91,161	44,350.30	3,547	12,011	6,788.30	125.26	447.03	219.12
5,000	23,615	76,870	43,315.50	4,086	18,449	7,244.70	312.31	955.73	550.42
10,000	25,322	113,933	57,053.70	4,098	25,665	9,383.30	717.07	2,852.37	1,482.95
Dcm-based bounding method									
10	4,348	38,008	11,682.80	928	16,534	3,903.50	0.07	0.67	0.20
20	10,334	135,627	42,372.90	3,518	50,889	14,189.70	0.35	4.70	1.46
50	10,009	63,998	28,842.10	2,706	10,653	6,057.80	0.85	5.43	2.44
100	26,202	126,039	69,249.00	5,218	46,734	19,825.60	4.43	21.35	11.72
200	39,972	141,281	69,207.90	9,098	24,792	15,131.00	13.50	47.62	23.35
500	84,410	879,358	258,658.30	9,687	229,944	46,271.50	71.15	740.73	217.86
1,000	51,124	588,205	185,917.70	11,504	142,366	32,350.20	86.70	991.01	313.44
2,000	92,079	243,985	145,699.50	12,715	34,554	22,735.80	312.32	822.62	492.63
5,000	83,212	273,440	157,018.10	14,070	57,043	25,126.70	716.48	2,315.28	1,336.59
10,000	88,832	420,086	202,536.40	16,210	87,786	31,930.20	1,561.79	7,138.21	3,472.83

Table 5 Computational results for experiment D in Table 1

N	Iterations			Max. triangles			Time (s)		
	Min.	Max.	Ave.	Min.	Max.	Ave.	Min.	Max.	Ave.
JOGO(2007) bounding method									
10	1,546	10,272	3,455.40	353	4,384	1,210.60	0.04	0.25	0.08
20	3,261	23,784	10,546.30	942	10,401	4,202.50	0.15	1.17	0.51
50	3,424	26,321	9,037.90	902	3,033	1,787.10	0.42	3.21	1.10
100	8,631	38,740	20,839.70	1,440	15,614	5,968.30	2.10	9.40	5.06
200	11,374	31,149	18,604.70	2,561	7,331	4,418.20	5.56	15.15	9.05
500	25,686	225,775	60,419.40	2,974	56,600	12,151.50	31.34	272.73	73.28
1,000	16,119	121,410	49,726.00	4,060	36,009	9,533.30	40.03	294.57	121.40
2,000	25,003	91,161	44,350.30	3,547	12,011	6,788.30	125.26	447.03	219.12
5,000	23,615	76,870	43,315.50	4,086	18,449	7,244.70	312.31	955.73	550.42
10,000	25,322	113,933	57,053.70	4,098	25,665	9,383.30	717.07	2,852.37	1,482.95
Dcm-based bounding method									
10	187	981	409.00	27	93	52.40	0.00	0.03	0.00
20	200	598	346.60	29	88	54.90	0.00	0.03	0.01
50	169	495	308.60	24	77	42.40	0.01	0.06	0.03
100	200	557	311.30	32	70	51.10	0.04	0.12	0.07
200	208	361	279.20	31	71	47.00	0.12	0.18	0.15
500	316	642	457.90	51	104	71.70	0.56	0.89	0.70
1,000	264	866	490.10	45	149	73.00	1.45	2.71	1.93
2,000	221	554	375.60	34	79	58.70	4.50	5.93	5.17
5,000	214	579	356.70	37	81	54.20	24.65	28.54	26.18
10,000	209	685	420.70	33	123	70.90	94.00	104.15	98.51

Table 6 Computational results for experiment E in Table 1

N	Iterations			Max. triangles			Time (s)		
	Min.	Max.	Ave.	Min.	Max.	Ave.	Min.	Max.	Ave.
JOGO(2007) bounding method									
10	7,590	191,606	90,873.30	3,138	78,502	37,112.90	0.20	5.28	2.49
20	670	511,323	130,306.00	214	251,744	60,708.40	0.04	27.46	7.02
50	418	104,870	31,748.90	58	41,895	12,497.20	0.06	14.00	4.23
100	163	271,540	62,316.10	33	113,022	25,752.30	0.04	72.81	16.58
200	275	94,917	26,174.30	75	47,849	11,692.20	0.18	50.78	13.96
500	853	333,568	81,006.20	135	143,653	34,609.90	1.42	440.35	107.47
1,000	3,546	133,319	39,303.60	1,132	75,639	18,829.20	10.42	355.70	105.47
2,000	691	93,329	24,589.30	220	45,317	10,874.40	8.31	500.03	135.11
5,000	5,744	36,640	13,439.10	2,334	16,416	6,069.40	105.25	516.76	207.22
10,000	285	67,258	11,445.70	67	32,693	5,335.10	123.50	1,895.34	418.85
Dcm-based bounding method									
10	187	981	409.00	27	93	52.40	0.00	0.03	0.00
20	200	598	346.60	29	88	54.90	0.00	0.03	0.01
50	169	495	308.60	24	77	42.40	0.01	0.06	0.03
100	200	557	311.30	32	70	51.10	0.04	0.12	0.07
200	208	361	279.20	31	71	47.00	0.12	0.18	0.15
500	316	642	457.90	51	104	71.70	0.56	0.89	0.70
1,000	264	866	490.10	45	149	73.00	1.45	2.71	1.93
2,000	221	554	375.60	34	79	58.70	4.50	5.93	5.17
5,000	214	579	356.70	37	81	54.20	24.65	28.54	26.18
10,000	209	685	420.70	33	123	70.90	94.00	104.15	98.51

Table 7 Computational results for experiment F in Table 1

N	Iterations			Max. triangles			Time (s)		
	Min.	Max.	Ave.	Min.	Max.	Ave.	Min.	Max.	Ave.
JOGO(2007) bounding method									
10	91	154	122.30	9	16	13.40	0.00	0.01	0.00
20	89	166	131.70	10	26	20.00	0.00	0.01	0.00
50	116	182	156.70	22	39	28.40	0.01	0.03	0.01
100	138	235	182.10	23	35	29.10	0.03	0.04	0.04
200	164	237	197.40	33	41	37.60	0.09	0.12	0.10
500	181	297	247.40	34	49	41.00	0.35	0.46	0.41
1,000	177	327	277.20	35	55	46.60	1.09	1.35	1.26
2,000	199	343	292.00	30	57	44.90	3.79	4.31	4.13
5,000	248	350	293.50	38	63	51.90	21.51	22.42	21.91
10,000	255	475	343.50	42	67	55.30	81.64	85.60	83.23
Dcm-based bounding method									
10	96	159	127.30	11	26	19.40	0.00	0.01	0.00
20	97	201	139.70	15	38	27.00	0.00	0.01	0.00
50	128	224	172.60	25	44	33.90	0.00	0.01	0.01
100	147	228	191.80	27	42	36.10	0.01	0.03	0.02
200	187	248	214.10	37	53	43.90	0.04	0.06	0.06
500	202	318	264.50	38	54	43.50	0.21	0.28	0.24
1,000	214	355	287.10	42	57	48.80	0.64	0.79	0.72
2,000	213	340	304.60	28	65	47.60	2.07	2.35	2.28
5,000	272	384	314.90	41	63	52.90	11.57	12.18	11.80
10,000	261	480	352.80	43	71	57.00	43.15	45.57	44.16

Table 8 Computational results for experiment G in Table 1

N	Iterations			Max. triangles			Time (s)		
	Min.	Max.	Ave.	Min.	Max.	Ave.	Min.	Max.	Ave.
JOGO(2007) bounding method									
10	136	402	220.90	20	53	30.00	0.00	0.01	0.00
20	132	601	209.50	18	62	27.80	0.00	0.03	0.00
50	133	410	181.50	20	55	28.30	0.01	0.03	0.01
100	116	230	165.60	20	35	25.40	0.03	0.04	0.03
200	117	235	161.60	19	33	24.60	0.06	0.09	0.07
500	116	186	135.80	17	30	22.60	0.26	0.31	0.27
1,000	94	173	125.20	19	29	21.60	0.81	0.93	0.86
2,000	102	125	111.30	18	22	19.90	2.98	3.06	3.01
5,000	103	150	117.70	18	33	23.10	17.46	17.82	17.58
10,000	83	183	115.80	18	35	23.30	67.81	69.40	68.36
Dcm-based bounding method									
10	132	401	215.20	18	53	29.30	0.00	0.01	0.00
20	125	601	202.50	18	61	26.70	0.00	0.01	0.00
50	125	411	174.00	18	56	26.80	0.00	0.01	0.00
100	109	222	159.30	18	34	24.00	0.00	0.03	0.01
200	113	228	154.80	17	32	23.40	0.03	0.04	0.03
500	111	184	129.90	17	28	21.40	0.09	0.12	0.10
1,000	88	164	118.40	17	28	20.90	0.29	0.34	0.31
2,000	92	121	106.10	16	21	18.70	1.07	1.10	1.09
5,000	99	147	112.50	18	30	21.50	6.29	6.46	6.34
10,000	80	175	110.50	16	33	21.70	24.40	25.06	24.61

Table 9 Computational results for experiment H in Table 1

N	Iterations			Max. triangles			Time (s)		
	Min.	Max.	Ave.	Min.	Max.	Ave.	Min.	Max.	Ave.
JOGO(2007) bounding method									
10	158	940	346.30	19	109	45.00	0.00	0.01	0.00
20	170	663	265.10	24	85	35.70	0.00	0.01	0.00
50	142	705	271.50	19	107	39.90	0.00	0.06	0.02
100	132	277	185.20	21	41	27.80	0.01	0.04	0.03
200	137	303	191.50	24	46	30.70	0.06	0.12	0.08
500	133	219	154.40	20	33	25.90	0.26	0.32	0.27
1,000	111	192	139.80	21	34	24.90	0.82	0.93	0.86
2,000	107	137	126.70	20	27	23.50	2.93	3.03	2.99
5,000	107	195	127.90	20	34	24.70	17.14	17.79	17.29
10,000	104	199	129.10	21	35	24.90	66.89	68.28	67.25
Dcm-based bounding method									
10	151	940	335.50	18	109	42.40	0.00	0.01	0.00
20	160	662	256.50	23	86	34.40	0.00	0.01	0.00
50	129	701	262.40	19	106	38.70	0.00	0.01	0.00
100	124	267	175.60	21	41	26.50	0.00	0.01	0.01
200	129	297	180.80	21	43	28.30	0.01	0.03	0.02
500	122	214	145.20	20	33	24.10	0.07	0.09	0.07
1,000	103	187	132.90	19	33	23.50	0.23	0.26	0.24
2,000	101	129	116.80	18	24	20.60	0.81	0.84	0.82
5,000	101	189	120.10	20	35	23.00	4.68	4.90	4.73
10,000	94	192	122.10	19	34	23.90	18.20	18.68	18.33

Table 10 Computational results for experiment I in Table 1

N	Iterations			Max. triangles			Time (s)		
	Min.	Max.	Ave.	Min.	Max.	Ave.	Min.	Max.	Ave.
JOGO(2007) bounding method									
10	104	333	182.30	16	38	22.40	0.00	0.01	0.00
20	102	238	147.50	14	31	19.60	0.00	0.01	0.00
50	105	212	135.30	14	29	18.50	0.00	0.01	0.01
100	97	231	131.40	14	32	19.50	0.01	0.04	0.02
200	94	129	111.00	14	21	16.60	0.04	0.07	0.06
500	94	194	136.50	15	28	20.50	0.26	0.34	0.29
1,000	86	162	110.60	14	25	17.60	0.85	0.98	0.90
2,000	85	133	100.50	14	23	17.50	3.14	3.31	3.20
5,000	77	209	107.10	15	38	19.80	18.53	19.60	18.77
10,000	70	216	112.00	14	44	22.80	72.67	75.03	73.35
Dcm-based bounding method									
10	98	332	180.90	15	38	22.20	0.00	0.01	0.00
20	101	235	145.70	13	30	19.00	0.00	0.01	0.00
50	103	212	133.70	14	29	18.00	0.00	0.01	0.00
100	95	231	128.80	14	32	18.80	0.00	0.01	0.01
200	87	129	109.10	14	21	16.30	0.01	0.03	0.02
500	87	194	134.20	15	27	20.10	0.09	0.14	0.11
1,000	85	161	109.40	13	25	17.50	0.32	0.37	0.34
2,000	84	130	99.20	14	21	17.30	1.18	1.25	1.20
5,000	74	209	106.30	15	38	19.70	6.90	7.40	7.02
10,000	69	216	110.70	13	44	22.60	27.04	28.10	27.35

Table 11 Computational results for experiment J in Table 1

N	Iterations			Max. triangles			Time (s)		
	Min.	Max.	Ave.	Min.	Max.	Ave.	Min.	Max.	Ave.
JOGO(2007) bounding method									
10	120	7,159	3,926.20	22	1,483	860.10	0.00	0.10	0.06
20	130	32,216	4,923.20	27	4,962	872.00	0.00	1.01	0.15
50	90	1,357	418.90	24	304	80.40	0.00	0.10	0.03
100	138	763	469.80	46	191	109.00	0.03	0.10	0.07
200	85	614	209.30	25	135	51.50	0.06	0.21	0.09
500	100	624	234.30	29	96	56.40	0.25	0.65	0.35
1,000	164	256	198.30	41	86	57.00	0.95	1.09	1.01
2,000	122	387	195.80	34	85	53.20	3.18	4.01	3.42
5,000	146	310	217.50	43	105	70.90	18.67	19.98	19.24
10,000	162	391	249.90	48	149	87.70	72.67	76.23	74.03
Dcm-based bounding method									
10	165	1,298,546	134,099.90	35	599,174	60,936.20	0.00	10.26	1.05
20	175	33,342	5,118.10	43	5,498	949.60	0.00	0.48	0.07
50	157	1,447	519.40	56	290	105.70	0.00	0.06	0.02
100	239	836	572.80	96	198	124.80	0.01	0.06	0.04
200	174	681	317.40	53	157	101.20	0.04	0.10	0.05
500	187	843	362.00	69	173	101.20	0.14	0.39	0.20
1,000	264	452	344.20	89	152	116.10	0.48	0.60	0.54
2,000	218	673	353.00	79	199	131.10	1.50	2.14	1.68
5,000	246	601	395.00	77	223	141.80	8.23	9.53	8.77
10,000	292	742	472.80	91	249	178.60	31.50	34.75	32.80

Table 12 Computational results for experiment K in Table 1

N	Iterations			Max. triangles			Time (s)		
	Min.	Max.	Ave.	Min.	Max.	Ave.	Min.	Max.	Ave.
JOGO(2007) bounding method									
10	146	334	214.50	18	49	30.50	0.00	0.01	0.00
20	133	311	232.60	23	43	32.10	0.00	0.01	0.00
50	171	308	232.50	25	67	34.60	0.01	0.03	0.01
100	151	310	209.40	22	42	31.50	0.03	0.06	0.04
200	169	314	237.80	28	50	35.00	0.07	0.12	0.10
500	172	245	205.10	25	43	34.30	0.31	0.37	0.33
1,000	164	257	193.30	24	36	29.80	0.95	1.10	1.00
2,000	156	257	193.90	22	38	29.90	3.26	3.57	3.38
5,000	146	247	183.80	25	39	31.30	18.37	19.15	18.66
10,000	106	234	163.70	24	37	31.30	70.40	72.45	71.32
Dcm-based bounding method									
10	113	302	179.10	16	50	30.20	0.00	0.01	0.00
20	144	368	197.70	22	48	31.90	0.00	0.01	0.00
50	147	660	225.90	24	78	33.40	0.00	0.03	0.00
100	127	211	177.10	23	41	30.60	0.00	0.03	0.01
200	139	264	198.70	25	60	34.00	0.03	0.04	0.03
500	163	214	182.50	24	48	32.90	0.12	0.14	0.13
1,000	143	236	170.60	21	36	27.80	0.34	0.42	0.37
2,000	142	243	169.30	19	33	25.80	1.20	1.35	1.25
5,000	127	236	171.50	23	41	29.90	6.76	7.14	6.91
10,000	127	232	159.30	21	32	27.30	26.14	26.89	26.36

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